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An information theory framework for the analysis of scene complexity

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Abstract

In this paper we present a new framework for the analysis of scene visibility and radiosity complexity. We introduce a number of complexity measures from information theory quantifying how difficult it is to compute with accuracy the visibility and radiosity in a scene. We define the continuous mutual information as a complexity measure of a scene, independent of whatever discretisation, and discrete mutual information as the complexity of a discretised scene. Mutual information can be understood as the degree of correlation or dependence between all the points or patches of a scene. Thus, low complexity corresponds to low correlation and vice versa. Experiments illustrating that the best mesh of a given scene among a number of alternatives corresponds to the one with the highest discrete mutual information, indicate the feasibility of the approach. Unlike continuous mutual information, which is very cheap to compute, the computation of discrete mutual information can however be quite demanding. We will develop cheap complexity measure estimates and derive practical algorithms from this framework in future work.

Keywords: rendering, radiosity, Monte Carlo, information theory, entropy, mutual information

1. Introduction

Complexity reflects "the difficulty of describing a system, the difficulty of reaching a goal, the difficulty of performing a task, and so on" [20]. In this paper, we introduce complexity measures, quantifying how difficult it is to compute with accuracy the visibility and radiosity in a scene. The quantities we propose can be interpreted as the degree of correlation or dependence between all the points or patches of a scene.

The study of 3D scene visibility, in which only the mutual visibility of surfaces in the scene is considered, is a first step towards the study of 3D radiosity complexity, in which also the illumination on the surfaces is taken into account. The study of 3D scene visibility complexity is however also an interesting topic on its own with potential applications in fields such as AI, robotics and architectural design.

The framework in this paper will enable us to analyze the difficulty of performing illumination computations using Monte Carlo radiosity algorithms [33, 9, 28, 29, 24, 3]. Monte Carlo radiosity algorithms are radiosity algorithms in which Monte Carlo Markov chain techniques are used in order to solve the radiosity problem. They are important because explicit form factor computation and storage is avoided, yielding more reliable radiosity algorithms that are suited to visualize much more complex scenes than possible with more traditional radiosity algorithms [16, 17, 6]. Potential applications of the complexity study presented in this paper include cost prediction for radiosity computations and the development of meshing strategies that are optimal in the sense that they will allow lowest computational error for a given amount of work.

The organization of this paper is as follows. In section 2, we describe the basic concepts of the framework for studying scene visibility and radiosity complexity. This framework is applied next (section 3) to the study of scene visibility complexity. Finally, in section 4, we show how some results obtained for scene visibility complexity can be also extended to take into consideration the illumination in a scene.

2. A framework for the study of scene visibility and radiosity complexity

2.1. Radiosity, form factors, visibility and random walks

2.1.1. Radiosity and form factors

The *radiosity equation* solves for the illumination in a diffuse environment. It can be written in the form

$$B(x) = E(x) + R(x)\int_S B(x')V(x,x')\frac{\cos\theta\cos\theta'}{\pi r^2}dA' \quad (1)$$

where $B(x)$ is the radiosity, $E(x)$ is the emittance, $R(x)$ is the reflectance, $\mathcal{S}$ is the set of surfaces that form the environment, x, x' are points on surfaces of the environment, dA' is an area differential at point x', r is the distance between x and x', $V(x,x')$ is a visibility function equal to 1 if x and x' are mutually visible and 0 otherwise, θ, θ' are the angles which the normals at x, x' form with the line joining them, and $V(x,x')\frac{\cos\theta\cos\theta'}{\pi r^2}$ is the differential form factor between x and x'.

To solve the radiosity equation we can use a finite element approach, and discretise the environment into n_p patches, considering the radiosities, emissivities and reflectances constant over the patches. In this way we transform the integral equation into the radiosity system of equations [16]:

$$B_i = E_i + R_i \sum_{j=1}^{n_p} F_{ij}B_j \quad (2)$$

where the *form factors* F_{ij} or coefficients of the radiosity system are only dependent on the geometry of the scene:

$$\begin{aligned} F_{ij} &= \frac{1}{A_i}\int_{A_i}\int_{A_j}\frac{\cos\theta_i\cos\theta_j}{\pi r^2}V_{ij}dA_idA_j \\ &= \frac{1}{A_i}\int_{A_i}\int_{\Omega}\frac{\cos\theta_i}{\pi}V_{ij}d\omega dA_i. \end{aligned} \quad (3)$$

Ω is the hemisphere on patch i and $d\omega$ is the differential of solid angle. We have the following properties for the form factors:

$$A_iF_{ij} = A_jF_{ji} \quad \forall i,j \quad (4)$$

$$\sum_{j=1}^{n_p} F_{ij} = 1 \quad \forall i \quad (5)$$

The form factor F_{ij} describes what fraction of the energy emitted by patch i will hit another patch j.

Consider a set of rays with origin uniformly distributed over the surface of i and directions distributed according to the cosine w.r.t. the surface normal on i. The form factor F_{ij} can be interpreted as the fraction of such lines that have j as the nearest patch intersected (see figure 1(a)). From integral geometry, one knows that such lines can also be derived from a uniform global distribution [28] (see figure 1(b)). The form factor F_{ij} can thus be considered as the probability of a line that exiting from or crossing i lands on j. If we identify the lines connecting two patches with visibility, the form factor will thus give us the *visibility* between patches [28, 31].

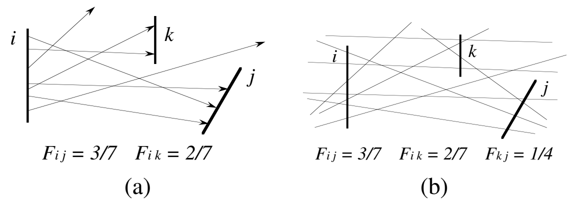


Figure 1: *Form factors can be computed with local (a) or global lines (b).*

2.1.2. Random walks

A *random walk* [27] in a scene can be considered as a *Markov chain* [23]. This is a discrete stochastic process defined over a set of states S which can be described by a *transition probability matrix P*. This matrix P has one row and one column for each state in S. The element $P_{j|i}$ in the matrix P is the probability that the next visited state will be j, given that the current state is i. Thus, for all $i, j \in S$, we have $\sum_{j=1}^{n_p} P_{j|i} = 1$, and paths are followed from state to state according to the given transition probabilities. A transition probability matrix is said to be aperiodic if it has no periodic state [7]. A periodic state is a state that can be visited back by a path starting from it only at multiples of a given period. A probability matrix is said to be irreducible if there is always a path between any two states.

The form factor matrix fulfills all required conditions to be a valid transition matrix of a random walk. The states of the random walk will correspond to the patches of a scene. In order to determine the next state of a random walk, the form factors with the current patch need to be sampled. Fortunately, such sampling can be carried out easily without having to compute the form factors explicitely [33, 26, 9, 30]. For the purpose of this paper we are mainly interested in the following two properties [31]:

- If the form factor matrix F is irreducible and aperiodic [7], then we have $\lim_{n\to\infty}(F^n)_{ij} \to \frac{A_j}{A_T}$ for all the n_p patches of a scene, where A_j is the area of patch j and $A_T = \sum_{i=1}^{n_p} A_i$.
  Thus, as the equilibrium distribution for a Markov chain is defined as the limit of the nth power of the transition matrix when n grows to infinity, if we know the areas of the patches, we also know the equilibrium distribution $\{\frac{A_i}{A_T}\}$ of the random walk. As a consequence, we have:
- When the length of a random path with transition matrix F grows to infinity, the number of hits on any patch i gets proportional to $\frac{A_i}{A_T}$, independently of where the path started its trajectory.

2.2. Complexity

Complexity is an active research area in many different fields. But, what is a complexity measure? W. Li's answer is[20]: "The meaning of this quantity should be very close to certain measures of difficulty concerning the object or the system in question: the difficulty of constructing an object, the difficulty of describing a system, the difficulty of reaching a goal, the difficulty of performing a task, and so on". There are many definitions of complexity corresponding with the different ways to quantify these difficulties and "there is not yet a consensus on a precise definition" [2].

In the two last decades, several complexity measures have been proposed from different fields to quantify the degree of structure or correlation of a system [12,21]. Feldman and Crutchfield [12] propose to call them "measures of *statistical complexity*". This is the concept of complexity that we have considered in this paper. Let us emphasize that this notion of complexity is different to computational complexity and to Kolmogorov-Chaitin complexity [22]. And concretely we will study statistical complexity from the view point of information theory.

In previous work[1,10], we applied the concept of entropy as defined in information theory to study of 3D scene visibility. We examined the relationship between the entropy of scene visibility and the expected value of the mean square *error* on the form factors when computed using random walk techniques as described above.

In this paper, we propose the application of other results from information theory to the problem of quantifying the *complexity* of 3D scenes. We consider two cases: 3D scene visibility complexity, when only visibility of the surfaces is taken into account, and 3D scene radiosity complexity, when also diffuse illumination is taken into account. This approach to scene complexity is different from the ones taken in [5], where the study was based on integral geometry results, in [18], where it was shown that the average number of intersections of the global lines with the objects of a scene was not a good complexity measure, and in [25], based on the reachability graph.

2.3. Information theory

In this section, we present some basic concepts of information theory [4,8]. The *Shannon entropy* H of a discrete random variable X with values in the set $\{a_1, a_2, \ldots, a_n\}$ is defined as

$$H(X) = -\sum_{i=1}^{n} p_i \log p_i \tag{6}$$

where $p_i = Pr[X = a_i]$, the logarithms are taken in base 2, and when $p_i = 0$ we take $p_i \log p_i = 0$. As $-\log p_i$ represents the *information* associated with the result a_i, the entropy gives the average information or the *uncertainty* of a random variable. The unit of information is called *bit*.

If we consider another random variable Y with probabilities q_i corresponding to values in the set $\{b_1, b_2, \ldots, b_m\}$, the *joint entropy* of X and Y is defined as

$$H(X,Y) = -\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij} \log p_{ij} \tag{7}$$

where $p_{ij} = Pr[X = a_i, Y = b_j]$, and the *conditional entropy* is defined as

$$H(X|Y) = -\sum_{j=1}^{m}\sum_{i=1}^{n} p_{ij} \log p_{i|j} \tag{8}$$

where $p_{i|j} = Pr[X = a_i | Y = b_j]$. The Bayes theorem expresses the relation between the different probabilities: $p_{ij} = q_j p_{i|j} = p_i p_{j|i}$. If X and Y are *independents*, we have $p_{ij} = p_i q_j$. The conditional entropy can be thought of in terms of a *channel* whose input is the random variable X and whose output is the random variable Y. $H(X|Y)$ corresponds to the uncertainty in the channel input from the receiver's point of view.

The *mutual information* between two random variables X and Y is defined as

$$I(X,Y) = H(X) - H(X|Y) \tag{9}$$

From the above definitions, we can obtain $I(X,Y) \geq 0$ and $I(Y,X) = I(X,Y)$. The mutual information represents the amount of information that one random variable, the output of the channel, gives about a second random variable, the input of the channel. $I(X,Y)$ is a measure of the shared information between X and Y.

The *relative entropy* or *Kullback-Leibler distance* between two probability distributions $p = \{p_i\}$ and $q = \{q_i\}$, that are defined over the same set $S = \{1, \ldots, n\}$, is defined as

$$D_{KL}(p\|q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \tag{10}$$

The relative entropy satisfies $D_{KL}(p\|q) \geq 0$, with equality only if $p = q$. It is also known as *discrimination* and it is not strictly a distance. Moreover, we want to emphasize that the mutual information can be expressed as

$$I(X,Y) = D_{KL}(\{p_{ij}\}\|\{p_i q_j\}) \tag{11}$$

The joint entropy of n random variables is defined as

$$\begin{aligned} H(X_1, \ldots, X_n) = H(X_1) + H(X_2|X_1) + \ldots \\ + H(X_n|X_1, \ldots, X_{n-1}) \end{aligned} \tag{12}$$

and the *entropy rate* or *entropy density* of a chain of random variables is defined by

$$\begin{aligned} h &= \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \ldots, X_n) \\ &= \lim_{n\to\infty} H(X_n|X_{n-1}, \ldots, X_1) \end{aligned} \tag{13}$$

representing the average information content per output symbol. It is the uncertainty associated with a given symbol if all the preceding symbols are known and can be viewed as the intrinsic *unpredictability* or the irreducible *randomness* associated with the chain" [14].

In particular, a Markov chain can be considered as a chain of random variables complying with

$$H(X_n|X_1,X_2,\ldots,X_{n-1}) = H(X_n|X_{n-1}) \tag{14}$$

An important result is the following theorem: A Markov chain with equilibrium distribution has entropy rate or information content

$$\begin{aligned} h &= \lim_{n\to\infty} \frac{1}{n} H(X_1,X_2,\ldots,X_n) \\ &= H(Y|X) = -\sum_{i=1}^{n} w_i \sum_{j=1}^{n} P_{j|i} \log P_{j|i} \end{aligned} \tag{15}$$

where w_i is the equilibrium distribution.

Finally, we define the *excess entropy* of an infinite chain as

$$E = \lim_{n\to\infty} (H(X_1,X_2,\ldots,X_n) - nh) \tag{16}$$

where h is the entropy rate of the chain and n is the length of this chain. The excess entropy can be interpreted as the mutual information between two semi-infinite halves of the chain.

3. Scene visibility complexity

In this section we study the complexity of a scene from the point of view of visibility, applying the basic concepts exposed in the above section.

3.1. Discrete mutual information

3.1.1. Definitions

We can consider [10] a random walk in a discretised scene as a Markov chain where $P_{j|i} = F_{ij}$, $n = n_p$, $w_i = \frac{A_i}{A_T}$ and X,Y are dependent random variables taking values over the set of patches $S = \{1,2,\ldots,n_p\}$ with $\{\frac{A_i}{A_T}\}$ as probability distribution. Thus, we define the *scene visibility entropy rate*, or simply *scene visibility entropy*, as

$$H_s = H(Y|X) = -\sum_{i=1}^{n_p} \frac{A_i}{A_T} \sum_{j=1}^{n_p} F_{ij} \log F_{ij} \tag{17}$$

It is important to emphasize that *a scene can be considered as a channel with input (source) X and output (destination) Y, where X and Y take values over the set of patches with* $\{\frac{A_i}{A_T}\}$ *as probability distribution and the channel transition matrix is the form factor matrix*. Thus, H_s measures the average uncertainty that remains about the destination patch when the source patch is known.

The Bayes theorem can be expressed by the property of the form factors

$$p_{ij} = \frac{A_i}{A_T} F_{ij} = \frac{A_j}{A_T} F_{ji} \tag{18}$$

and from this we obtain $H(Y|X) = H(X|Y)$, and thus the symmetry or reversibility of the channel is shown. Also, we define the *positional entropy* as

$$H_p = H(X) = H(Y) = -\sum_{i=1}^{n_p} \frac{A_i}{A_T} \log \frac{A_i}{A_T} \tag{19}$$

and may be interpreted as the uncertainty on the position (patch) of a particle travelling an infinite random walk.

The *discrete scene visibility mutual information* is defined as

$$\begin{aligned} I_s &= I(X,Y) = H(Y) - H(Y|X) \\ &= -\sum_{i=1}^{n_p} \frac{A_i}{A_T} \log \frac{A_i}{A_T} + \sum_{i=1}^{n_p} \frac{A_i}{A_T} \sum_{j=1}^{n_p} F_{ij} \log F_{ij} \end{aligned} \tag{20}$$

and can be interpreted as the amount of information that the destination patch conveys about the source patch, and vice versa. Consequently, I_s *is a measure of the average information transfer in a scene*. It is interesting to remember that mutual information can be defined as a Kullback-Leibler distance (section 2.3):

$$I(X,Y) = D_{KL}(\{p_{ij}\} \| \{p_i q_j\})$$

Thus, I_s is the "distance" or discrimination between the scene probability distribution $\{p_{ij}\} = \{\frac{A_i}{A_T} F_{ij}\}$ and its independence distribution $\{p_i q_j\} = \{\frac{A_i A_j}{A_T^2}\}$ and can be expressed as

$$\begin{aligned} I_s &= \sum_{i=1}^{n_p} \sum_{j=1}^{n_p} \frac{A_i F_{ij}}{A_T} \log \frac{\frac{A_i F_{ij}}{A_T}}{\frac{A_i A_j}{A_T^2}} \\ &= \sum_{i=1}^{n_p} \sum_{j=1}^{n_p} \frac{A_i F_{ij}}{A_T} \log \frac{F_{ij} A_T}{A_j} \end{aligned} \tag{21}$$

Consequently, discrete mutual information can be interpreted as a "distance" to independence.

3.1.2. Discussion

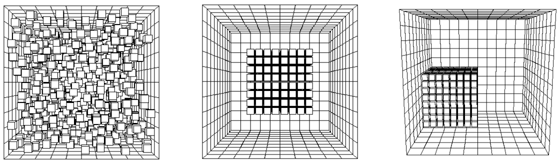
In a 3D scene context, it is especially interesting to ask about the extremal cases of maximum and minimum visibility entropy, which correspond to the maximum disorder (unpredictability or randomness in the ray path) and the maximum order (predictability), respectively. We must remark here that the concepts of order and disorder are not directly referred to the collocation of objects in space, but to visibility criteria. Maximum unpredictability is obtained in scenes with no privileged visibility directions, and maximum predictability in the contrary case.

Both cases can be illustrated with the following two examples:

- The maximum entropy is exemplified by the interior of an empty sphere divided into equal area patches. Here all form factors are equal and the uncertainty of the destination patch is maximum (no visibility direction is privileged). The information transfer is zero: $I_s = 0$. The sphere represents equilibrium or uniformity and can be considered as a channel where the variables X and Y are independent, because in a sphere $F_{ij} = \frac{A_j}{A_T}$ and thus $p_{ij} = \frac{A_i}{A_T} F_{ij} = \frac{A_i}{A_T}\frac{A_j}{A_T} = p_i p_j$. This is the expression of *independence* in a scene. Thus, if independence is represented by a sphere, discrete mutual information expresses the "distance" between a discretised scene and a discretised sphere with the same number of patches and the same area distribution of the patches.

- The minimum entropy is represented by a scene with almost touching objects, as for instance two very nearby concentrical spheres which are discretised in an identical manner. There are strongly privileged visibility directions (for each patch one form factor is near 1). The information transfer is maximum.

In scenes with the same discretisation (as in figure 2, where we have a cubical enclosure with 512 interior cubes), and consequently with the same H_p, we can observe (table 1) that the increase of H_s remains compensated for a decrease of I_s, and vice versa.



(a) (b) (c)

Figure 2: *Random (a) and clustered configurations (b,c) with 512 cubes.*

Scene	*Lines*	H_s	I_s	$E(MSE)$
Fig.2a	10^6	6.370	5.171	1284.6
Fig.2a	10^7	6.761	4.779	1286.7
Fig.2b	10^6	5.072	6.469	949.3
Fig.2b	10^7	5.271	6.270	950.1
Fig.2c	10^6	4.674	6.867	898.8
Fig.2c	10^7	4.849	6.692	900.9

Table 1: *Results for the three cubical enclosures with 512 small cubes (figures 2a, 2b and 2c). The $E(MSE)$ is normalized to a single line. For each scene, $H_p = 11.541$ and 10^6 and 10^7 lines have been cast.*

What is the accuracy of the values presented in the different tables? The error in these values, which is manifested when the number of casting lines is increased, is directly related to error in form factor computation. In this paper, the form factors have been computed with the Monte Carlo method described in [28, 31]. If we study [10] the relationship between the entropy H_s and the mean square error of all form factors $E(MSE)$:

$$E(MSE) \;=\; \frac{1}{N}(n_p - \sum_i \sum_j F_{ij}^2) \qquad (22)$$

where N is the total lines cast, we can observe (table 1) that the larger the entropy the larger the error in form factors computation. This means that, for a given computational error, we need to cast more lines for a scene with more entropy. It is also shown that the increase in the number of lines increases the entropy estimation and, logically, decreases the mutual information estimation.

3.1.3. Complexity proposal

Frequently, the concept of entropy has been used as a starting point in order to study complexity [12, 13, 21, 34, 35]. But, from the viewpoint of statistical complexity, entropy, although related to complexity, is not accepted as an adequate measure of the complexity of a system [12, 34]: "it has become more broadly understood that a system's randomness and unpredictability fail to capture its patterns and correlational structure" [13].

Mutual information however, which expresses the information transfer between variables or the average information conveyed by a random walk, has been proposed [20, 34] as a complexity measure and, according to W. Li [19], is considered as a correlation measure of a system. On the other hand, the excess entropy, which has been given other names such as "stored information" or "effective measure complexity" [14], is probably the most accepted complexity measure from the information theory approach and it is a form of mutual information. In our case, mutual information is equal to excess entropy:

Proposition 1 In the context of scene visibility, the excess entropy turns into the mutual information.

Proof: From (12) and (14), we have

$$\begin{aligned} H(X_1,\ldots,X_n) &= H(X_1)+\ldots+H(X_n|X_1,\ldots,X_{n-1}) \\ &= H(X_1)+H(X_2|X_1)+\ldots+H(X_n|X_{n-1}) \\ &= H_p+(n-1)H_s \end{aligned}$$

Thus

$$E = \lim_{n\to\infty}(H_p+(n-1)H_s - nH_s) = H_p - H_s = I_s \quad \square$$

This proposition and the meaning of mutual information enable us to suggest that *mutual information of scene visibility can be considered as a complexity measure of scene visibility*.

3.2. Continuous mutual information

3.2.1. Definitions

By discretising a scene into patches, a distortion or error is introduced. In a way, to discretise means to equalize. Obviously, the maximum accuracy of the discretisation is accomplished when the number of patches tends to infinity. How does mutual information behave in that limit case?

According to information theory [8], mutual information between two continuous random variables X and Y is the limit of the mutual information between their discretised versions. Entropy of a continuous random variable however does not equal the entropy of the discretised random variable in the limit of finer discretisation [14]. Thus, in our case, discrete mutual information I_s converges to continuous mutual information I_s^c when the number of patches tends to infinity: $I_s^c = \lim_{n_p \to \infty} I_s$. Below we will see that this fact is very important for us because it will enable us to calculate a relative distance to the ideal discretisation, represented by the continuous mutual information. Scene visibility entropy however tends to infinity when the number of patches tends to infinity.

We will obtain the continuous expression for all formulae using the following substitutions:

- Each summatory by an integral. For instance, $\sum_{i=1}^{n_p} \sum_{j=1}^{n_p} F_{ij}$ by $\int_{x \in S} \int_{y \in S} F(x,y)dxdy$, where S stands for the surfaces in the scene.
- $\frac{A_i}{A_T}$ by $\frac{1}{A_T}$. This means to substitute the discrete probability of taking patch i by the continuous probability of selecting any point.
- F_{ij} by $F(x,y)$. This means to substitute a patch-to-patch form factor by a point-to-point one. Remember that the value of $F(x,y)$ is $\frac{cos\theta_x cos\theta_y}{\pi d(x,y)^2}$ for mutually visible points, zero otherwise, being θ_x and θ_y the angles which the normals at x, y form with the segment joining x and y, and $d(x,y)$ the distance between x and y.

Thus, the above discrete formulae convert into the following:

$$H_p^c = -\int_{x \in S} \frac{1}{A_T} \log \frac{1}{A_T} dx = \log A_T \quad (23)$$

$$H_s^c = -\int_{x \in S} \int_{y \in S} \frac{1}{A_T} F(x,y) \log F(x,y) dxdy \quad (24)$$

$$\begin{aligned} I_s^c &= \log A_T + \int_{x \in S} \int_{y \in S} \frac{1}{A_T} F(x,y) \log F(x,y) dxdy \\ &= \int_{x \in S} \int_{y \in S} \frac{1}{A_T} F(x,y) \log(A_T F(x,y)) dxdy \end{aligned} \quad (25)$$

where H_p^c is the continuous positional entropy, H_s^c is the continuous scene visibility entropy and I_s^c is the continuous scene visibility mutual information. In the particular case of a sphere, as any pair (x,y) fulfills $F(x,y) = \frac{1}{A_T}$, the result obtained is, as expected, $I_s^c = 0$. Remember that in a sphere $I_s = 0$ and thus $\lim_{n_p \to \infty} I_s = 0$. Next, we will pay attention to continuous mutual information. Continuous entropy will be studied in future work.

3.2.2. Computation

The continuous mutual information integral can be easily solved by Monte Carlo integration. Let us reparametrize the integral:

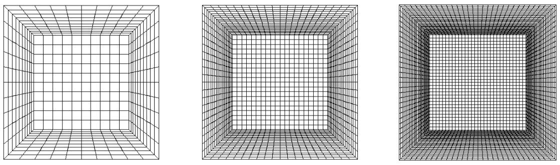
$$\begin{aligned} I_s^c &= \int_{x \in S} \int_{y \in S} \frac{1}{A_T} F(x,y) \log(A_T F(x,y)) dxdy \\ &= \int_{x \in S} \int_{\omega \in \Omega} \frac{1}{\pi A_T} cos\theta_x log(A_T F(x, y(x,\omega))) dxd\omega \end{aligned} \quad (26)$$

where $y(x,\omega)$ is the point visible from x in the ω direction. We will use now $\frac{1}{\pi A_T} cos\theta_x$ as probability density function (we are considering here as integration variables x and ω). It is easy to check that $\int_{x \in S} \int_{\omega \in \Omega} \frac{1}{\pi A_T} cos\theta_x dxd\omega = 1$. Drawing samples according to this distribution means simply selecting first a random point in the scene upon the area and a direction upon the form factor distribution. This can be achieved with *local* lines or, easier, with *global* lines. This is because the global lines are naturally distributed upon areas and form factors. The result obtained is

$$\begin{aligned} I_s^c &\approx \frac{1}{N} \sum_{k=1}^{N} \log(A_T F(x_k, y_k(x_k, \omega_k)) \\ &= \frac{1}{N} \sum_{k=1}^{N} \log(\frac{A_T cos\theta_x cos\theta_y}{\pi d(x,y)^2}) \end{aligned} \quad (27)$$

In the global line case, N stands for the total number of pairs of points considered, which is the total number of intersections divided by two.

As we can see in tables 2 and 3, corresponding to figure 3 and 2(a) respectively, the computation cost of I_s^c is much lower than the one of I_s: with few lines I_s^c can be computed with enough precision, unlike I_s which needs a lot of lines to get a precise measurement.



(a) (b) (c)

Figure 3: *Three empty cubical enclosures with their faces discretised into (a) 10x10, (b) 20x20 and (c) 30x30 patches.*

3.2.3. Discussion

The following proposition will enable us to analyze better the relationship between I_s^c and I_s:

Scene	*Lines*	H_s	I_s	I_s^c
Fig.3a	10^5	6.482	2.747	1.610
Fig.3a	10^7	7.821	1.408	1.612
Fig.3a	10^9	7.838	1.391	1.610
Fig.3b	10^5	5.342	5.887	1.608
Fig.3b	10^7	9.420	1.809	1.611
Fig.3b	10^9	9.731	1.498	1.610
Fig.3c	10^5	4.313	8.086	1.610
Fig.3c	10^7	9.684	2.715	1.611
Fig.3c	10^9	10.852	1.547	1.610

Table 2: *Results for the cubical enclosure with different discretisations of its faces (figure 3a:10x10, figure Fig.3b:20x20 and figure Fig.3c:30x30). For each scene,* 10^5, 10^7 *and* 10^9 *lines have been cast.*

Lines	10^4	10^5	10^6	10^7
I_s	8.773	6.398	5.171	4.779
I_s^c	5.650	5.636	5.631	5.632

Table 3: *Results for the random configuration with 512 cubes (figure 2a).*

Proposition 2 The continuous scene visibility mutual information is the least upper bound to the discrete scene visibility mutual information.

Proof: We know that $I_s^c = \lim_{n_p \to \infty} I_s$. Thus, we must only show that I_s^c is an upper bound to I_s. Let us imagine a discretised scene with discrete mutual information I_s. It is sufficient to show that, if any patch is divided into two patches, the discrete mutual information I_s' of the new scene fulfills $I_s' - I_s \geq 0$. This can be proved [11] with the properties of the form factors and the concavity of the logarithm function for non-negative numbers. $\square$

In conclusion, continuous mutual information I_s^c, which is independent of the discretisation, expresses with maximum accuracy the information transfer or correlation in a scene. This is an *absolute* measure of the complexity of scene visibility. On the other hand, discrete mutual information I_s expresses the complexity of a discretised scene, which is always lower than the corresponding I_s^c.

As an example, I_s^c of platonic solids and Cornell box (see figure 10) has been computed. In table 4, we can observe that the minimum complexity corresponds to a sphere and the maximum complexity to a tetrahedron. As we expected, the polyhedra that are nearer to the sphere (independence) are less complex (less correlation). Thus, complexity appears to be inversely proportional with the number of faces. I_s^c of Cornell box is clearly greater than just the empty cube, as we have increased its complexity by introducing objects in its interior.

Also, in table 5, we show the complexity for the scenes of figure 4. In figure 4a, an object formed by a table and four chairs is situated in the middle of a room. In figures 4b and 4c, arrays of such objects have been situated in the middle of the same room. In figures 4d, 4e and 4f, the same 16 objects have been distributed in different ways. We can see that the introduction of objects increases the complexity and that the scenes with the same objects (4c, 4d, 4e and 4f) show similar complexities. In this case, the increase of complexity is produced when there are objects near the walls because this fact increases correlation in the scene.

Scene	I_s^c
sphere	0
icosahedron	0.5428
dodecahedron	0.8254
octahedron	1.2583
cube	1.6093
tetrahedron	2.6227
Cornell box	2.800

Table 4: *CMI of platonic solids and Cornell box. For each scene,* 10^6 *lines have been cast.*

Scenes	*4a*	*4b*	*4c*	*4d*	*4e*	*4f*
I_s^c	3.837	4.102	5.023	5.043	5.044	5.089

Table 5: *Continuous mutual information for the scenes of figure 4. For each scene,* 10^6 *lines have been cast.*

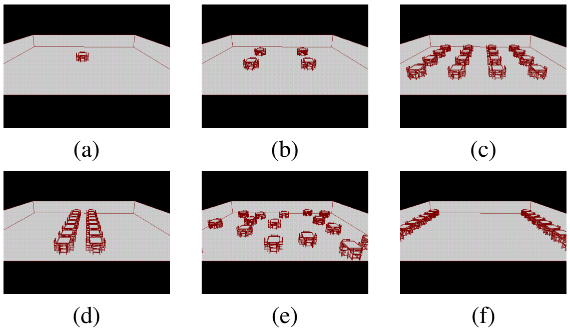


Figure 4: *An object formed by a table and four chairs (a) and an array of 2x2 objects with the same composition (b) have been situated in the middle of a room. Also, the same 16 objects have been distributed in four different ways (c, d, e and f).*

3.3. Discretisation accuracy

Proposition 2 suggests that the ratio of continuous and discrete mutual information may yield information about the error which occurs in the discretisation process and the difficulty in getting a precise discretisation. We make two fundamental proposals which will be contrasted with the results of various sample scenes:

- From the fact that the ideal discretisation, represented by I_s^c, is the one that captures all the information transfer in a scene, we can confirm that between different discretisations of the same scene the most precise one will be the one that has a higher I_s, i.e., the one that best captures information transfer. From this statement, we express the *discretisation accuracy* as the quotient $\frac{I_s}{I_s^c}$ and the *discretisation relative error* as the quotient $\frac{I_s^c - I_s}{I_s^c}$.
- We conjecture that I_s^c expresses the difficulty of discretisation. The higher the I_s^c, i.e., when there is more information transfer in a scene, the more difficult it is to obtain an accurate discretisation and probably more refinements will be necessary to achieve a given precision. According to this, the difficulty in discretising a sphere is null. And the polyhedra that are nearer to the sphere are less complex and so easier to discretise.

These proposals can be analyzed from the results shown in tables 6 and 7 and on the graphics of figures 7 and 8, which have been obtained from figures 5 and 6. Initially, 64 cubes are grouped very closely together in the center of the cubical enclosure. Little by little they are separated and moved outwards until they almost touch the walls. Complexity has been calculated for this sequence of scenes. Figure 7, showing continuous versus discrete mutual information, indicates that according to our definitions the more complex scenes are those that have surfaces closer to one another (figures a,f and b).

This sequence of scenes has been discretised in two different ways with the same number of patches. In the first sequence (figure 5), the discretisation of the cubes is finer whereas in the second (figure 6) the discretisation of the walls is finer. The accuracy of the discretisation appears to be higher in the "middle" scenes (b, c, d and e) and lower in the "extremal" scenes (a and f). Scenes a and f are the most complex scenes. Consequently, these scenes should have a finer discretisation in order to obtain greater accuracy. In other words, these scenes are the ones most difficult to discretise. Comparing alternative discretisations (figure 5a and 6a) for the scene shown in figure 8, the best discretisation appears to correspond to 5a because the discretisation is finer in the narrow spaces between the cubes. In contrast, when the cubes are near the walls, greater precision is obtained when the discretisation of the walls is finer.

These experiments suggest that discretisation accuracy may be used to choose a better discretisation from several alternatives and, while computational error is deeply related to entropy, discretisation error is related to mutual information. In future work, the relationship between both kinds of errors will be studied.

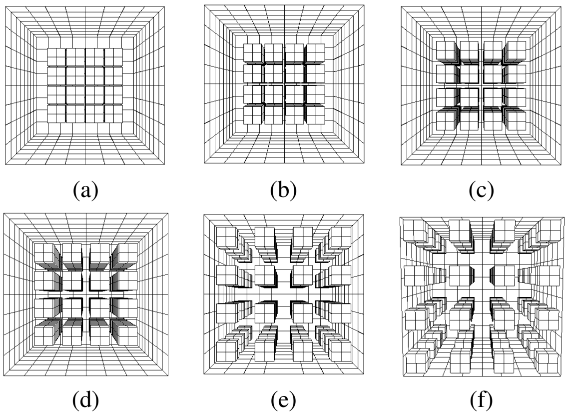


Figure 5: *64 cubes are grouped very closely together in the center of the cubical enclosure and they are separated and moved outwards until they almost touch the walls. The discretisation of the cubes (1536 patches) is finer than the one of the walls (384 patches).*

4. Towards scene radiosity complexity

So far, we have only considered visibility of a scene. In this section we will make a leap forward and will set the basis for the study of radiosity complexity.

4.1. Transition matrix for radiosity

The research on visibility presented in the previous section has been based on the existence of a Markov chain (form

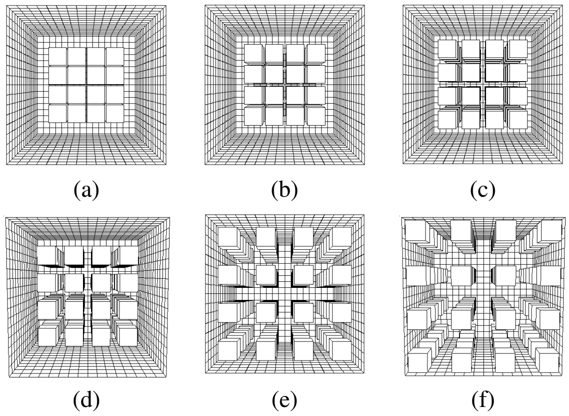


Figure 6: *64 cubes are grouped very closely together in the center of the cubical enclosure and they are separated and moved outwards until they almost touch the walls. The discretisation of the walls (1536 patches) is finer than the one of the cubes (384 patches).*

Scene	Fig.5a	Fig.5b	Fig.5c	Fig.5d	Fig.5e	Fig.5f
I_s	5.492	5.054	4.672	4.395	4.356	4.775
I_s^c	6.430	5.678	5.177	4.867	5.015	6.055
I_s/I_s^c	0.854	0.890	0.902	0.903	0.869	0.789

Table 6: *Results for the scenes of figure 5. For each scene,* 10^8 *lines have been cast.*

Scene	Fig.6a	Fig.6b	Fig.6c	Fig.6d	Fig.6e	Fig.6f
I_s	5.110	4.809	4.543	4.348	4.483	4.932
I_s^c	6.430	5.678	5.177	4.867	5.015	6.055
I_s/I_s^c	0.795	0.847	0.878	0.893	0.894	0.814

Table 7: *Results for the scenes of figure 6. For each scene,* 10^8 *lines have been cast.*

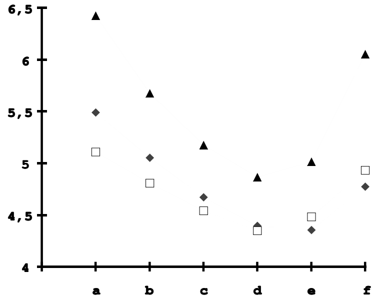


Figure 7: *Continuous (triangles) and discrete mutual information in vertical axis for the scenes of figures 5 (diamonds) and 6 (squares).*

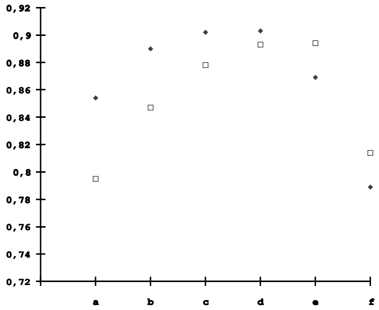


Figure 8: *Discretisation accuracy in vertical axis for the scenes of figures 5 (diamonds) and 6 (squares).*

factors) and the knowledge of its equilibrium distribution. Thus, to study the complexity of a scene *with* illumination, we need to find an analog of the form factor matrix for the radiosity setting. This analog appears naturally when the null variance probability transition matrix

$$p_{j|i} = \frac{R_i F_{ij} B_j}{B_i - E_i}$$

is considered [32]. This matrix corresponds to the transition probabilities that lead to null variance estimators. The null variance matrix must have a preferred position between the different possible transition matrices. To obtain the equilibrium distribution is not difficult. Using the left eigenvalue property [7], we obtain (without normalization)

$$p_i = A_i \frac{(B_i - E_i)}{R_i} B_i = A_i B_i^{in} B_i^{out} \qquad (28)$$

where $B_i^{in} = \frac{(B_i - E_i)}{R_i}$ is incoming radiosity and $B_i^{out} = B_i$ is the outgoing one. It is immediate to check that these probabilities fulfill $p_i p_{j|i} = q_j p_{i|j}$, this is

$$A_i \frac{(B_i - E_i)}{R_i} B_i \frac{R_i F_{ij} B_j}{B_i - E_i} = A_j \frac{(B_j - E_j)}{R_j} B_j \frac{R_j F_{ji} B_i}{B_j - E_j} \qquad (29)$$

which is an extended reciprocity relation (see equation (4)). Thus, the analogy is complete.

The entropy, mutual information and other quantities can be defined straightforwardly for the radiosity setting using this analogy. We define

$$\mathcal{A}_i = A_i \frac{(B_i - E_i)}{R_i} B_i \tag{30}$$

$$\mathcal{A}_T = \sum_i \mathcal{A}_i \tag{31}$$

$$\mathcal{F}_{ij} = \frac{R_i F_{ij} B_j}{B_i - E_i} \tag{32}$$

These definitions can be interpreted as a mapping of a given scene into a new (imaginary) scene, transforming the areas and the visibility channels according to formulae (30). Studying the radiosity complexity of the original scene corresponds to studying the visibility complexity of the new scene. According to section 3, the entropy and mutual information are

$$\mathcal{H}_p = -\sum_{i=1}^{n_p} \frac{\mathcal{A}_i}{\mathcal{A}_T} \log \frac{\mathcal{A}_i}{\mathcal{A}_T} \tag{33}$$

$$\mathcal{H}_s = -\sum_{i=1}^{n_p} \frac{\mathcal{A}_i}{\mathcal{A}_T} \sum_{j=1}^{n_p} \mathcal{F}_{ij} \log \mathcal{F}_{ij} \tag{34}$$

$$\mathcal{I}_s = \sum_{i=1}^{n_p} \sum_{j=1}^{n_p} \frac{\mathcal{A}_i \mathcal{F}_{ij}}{\mathcal{A}_T} \log \frac{\frac{\mathcal{A}_i \mathcal{F}_{ij}}{\mathcal{A}_T}}{\frac{\mathcal{A}_i \mathcal{A}_j}{\mathcal{A}_T^2}} \tag{35}$$

4.2. Discussion

As a first example, let us consider the case where the resulting radiosity is constant for all patches. This should not introduce any complexity to the visibility case, thus the transformation considered above should result into the identity transformation. To obtain constant radiosity B everywhere, it is easily found that we must have $B = R_i B + E_i$, for all i. But, then $B_i^{in} = \frac{B - E_i}{R_i} = \frac{R_i B}{R_i} = B$, and this means that

$$\frac{\mathcal{A}_i}{\mathcal{A}_T} = \frac{A_i B^2}{A_T B^2} = \frac{A_i}{A_T} \tag{36}$$

$$\mathcal{F}_{ij} = \frac{R_i F_{ij} B_j}{B_i - E_i} = F_{ij} \tag{37}$$

We have computed the visibility and radiosity complexity for the labyrinth scene (see figure 9), with $R_i + E_i = 1$ for each patch, and thus radiosity is equal to 1 everywhere. The results shown in table 8 confirm the theoretical prediction.

As a second experiment, we have computed the radiosity complexity of the boxes scene shown in figure 10. Six different discretisations have been generated: three rough and three finer ones. Each rough discretisation contains almost the same number of patches and so does each fine discretisation. Three meshing strategies have been tried out: quality triangulation yielded a uniform mesh, hierarchical refinement radiosity with refinement based on transported power as well as smoothness of received radiosity yielded adaptively refined meshes. The continuous mutual information (not computed) is identical in all cases. The discrete mutual information in table 9 reflects that a finer mesh is indeed a better mesh. Among equally fine meshes, the uniform mesh is quantified to be the worst.

	$\mathcal{H}_p$	$\mathcal{H}_s$	$\mathcal{I}_s$
Visibility	10.6883	6.47804	4.21022
Radiosity	10.6842	6.47833	4.20591

Table 8: *Results for the labyrinth scene of figure 9, with constant radiosity everywhere. For this case both visibility and radiosity complexity are the same.*

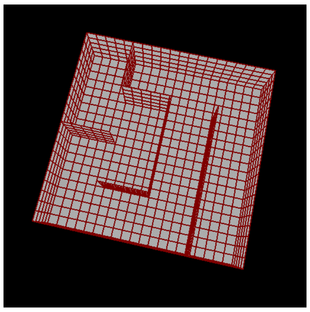


Figure 9: *Labyrinth scene used to show the equivalence of visibility and radiosity complexity when radiosity is constant everywhere (see table 8).*

Unlike previously developed error estimates in radiosity, discrete radiosity mutual information reflects both computational and discretisation error at the same time. The precise interplay between these two important sources of error in the discrete radiosity mutual information however remains to be investigated in more detail.

5. Conclusions and future research

We have presented in this paper an information theory approach for the analysis of scene visibility and radiosity complexity. The measures we propose for complexity are continuous and discrete mutual information. We have proved that continuous mutual information is the least upper bound of the discrete one. Discrete mutual information gives the information transfer or correlation in a discretised scene and, thus, continuous mutual information is the maximum pos-

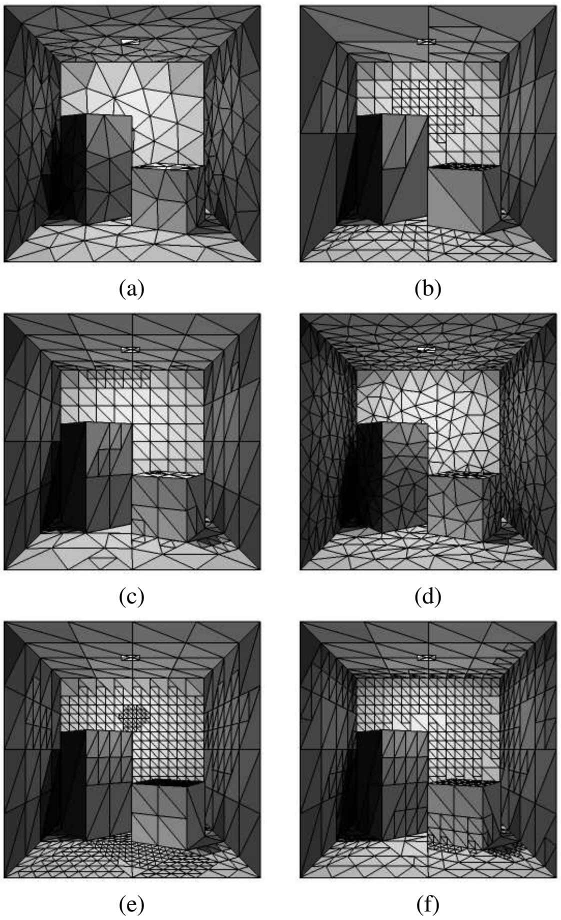


Figure 10: *Cornell box scene with different discretisations used to show the increase of mutual information for a finer and more accurate meshing. Results are in table 9.*

Fig.	*Mesh*	*Patches*	*Lines*(10^6)	$\mathcal{H}_p$	$\mathcal{H}_s$	$\mathcal{I}_s$
a	uniform	464	16.5	6.94	2.52	4.42
b	power	453	16.5	8.46	2.59	5.87
c	smooth	459	16.5	7.51	2.56	4.94
d	uniform	1146	41.2	8.30	2.75	5.55
e	power	1143	41.2	9.80	2.82	6.98
f	smooth	1146	41.2	8.98	2.80	6.18

Table 9: *Results for the Cornell box scene (figure 10) with different discretisations.*

sible information transfer or correlation. On the other hand, continuous mutual information also expresses how difficult it is to discretise a scene to compute with accuracy the visibility and radiosity, and discrete mutual information gives us a measure of how well we have done it. In this direction we have shown that the best discretisation into equal number of patches is the one with higher discrete mutual information.

Although continuous mutual information is very cheap to compute, a drawback for the practical use of our results is the high computing cost of an accurate value for discrete mutual information. We will look then in our future work for cheaper alternatives. A possible alternative is the *distance to independence*, a complexity measure introduced by Solé et al. [35].

The precise interplay between computational and discretisation error in the discrete radiosity mutual information remains to be investigated in more detail. This paper is a first step, paving the way for the development of practical algorithms and strategies for cost prediction and optimal meshing, which will be undertaken in our future work.

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